STABILITY OF RELATIVISTIC MATTER WITH MAGNETIC FIELDS FOR NUCLEAR CHARGES UP TO THE CRITICAL VALUE

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Abstract. We give a proof of stability of relativistic matter with magnetic fields all the way up to the critical value of the nuclear charge $Z\alpha = 2/\pi$.

1. Introduction

We shall give a proof of the 'stability of relativistic matter' that goes further than previous proofs by permitting the inclusion of magnetic fields for values of the nuclear charge Z all the way up to $Z\alpha = 2/\pi$, which is the well known critical value in the absence of a field. (The dimensionless number $\alpha = e^2/\hbar c$ is the 'fine-structure constant' and equals $1/137.036$ in nature.) More precisely, we shall show how to modify the earlier proof of Theorem 2 in [LY] so that an arbitrary magnetic field can be included. Reference will freely be made to items in the [LY] paper.

The quantum mechanical Hamiltonian used here and in [LY], as well as the definition of stability of matter, will be given in the next section. For a detailed overview of this topic, we refer to [L1, L2]. For the present we note that stability requires a bound on α in two ways. One is the requirement, for any number of electrons, that $Z\alpha \leq 2/\pi$. In fact, if $Z\alpha > 2/\pi$ the Hamiltonian is not bounded below even for a single electron. The other requirement is a bound on α itself, $\alpha \leq \alpha_c$, even for arbitrarily small $Z > 0$, which comes into play when the number of particles is sufficiently large. It is known that $\alpha_c \leq 128/15\pi$; see [LY, Thm. 3] and also [DL].

For values of $Z\alpha$ strictly smaller than the critical value $2/\pi$, it has been shown that stability holds with a magnetic field included. This is the content of Theorem 1 in [LY], in which the critical value of α_c goes to zero as $Z\alpha$ approaches $2/\pi$, however. (The result in [LY, Theorem 1] does not explicitly include a magnetic field, but the fact that the proof can easily be modified was noted in [LLoSo].) A similar result, by a different method, was proved in [LLoSi].

The more refined Theorem 2 in [LY] gives stability for the 'natural' value $Z\alpha \leq 2/\pi$ and all $\alpha \leq 1/94$. While the true value of α_c is probably closer to 1, the value $1/94 > 1/137$ is sufficient for physics. The problem with the proof of [LY, Theorem 2] is that it does not allow for the inclusion of magnetic fields. Specifically, Theorems

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9–11 have to be substantially modified, and doing so was an open problem for many years. This will be accomplished here at the price of decreasing α_c from $1/94$ to $1/133$. Fortunately, this is still larger than the physical value $1/137$!

In a closely related paper [FLSe] we also show how to achieve a proof of stability for all $Z\alpha \leq 2/\pi$ with an arbitrary magnetic field, but the value of α_c there is very much smaller than the value obtained here. In particular, the physical value of $\alpha = 1/137$ is not covered by the result in [FLSe]. The focus of [FLSe] is much broader than 'stability of matter', however. It is concerned with a general connection between Sobolev and Lieb-Thirring type inequalities, and includes as a special case Theorem 4.5 of this paper. The proof of the general result in [FLSe] is much more involved than the one of the special case presented here, and yields a worse bound on the relevant constant.

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2. Definitions and Main Theorem

We consider N electrons of mass $m \geq 0$ with q spin states ($q = 2$ for real electrons) and K fixed nuclei with (distinct) coordinates $R_1, \ldots, R_K \in \mathbb{R}^3$ and charges $Z_1, \ldots, Z_K > 0$. The electrons interact with an external, spatially dependent magnetic field $B(x)$, which is given in terms of the magnetic vector potential $A(x)$ by $B = \mathrm{curl} A$. A pseudo-relativistic description of the corresponding quantum-mechanical system is given by the Hamiltonian

$$H_{N,K} := \sum_{j=1}^{N} \left(\sqrt{(p_j + A(x_j))^2 + m^2} - m \right) + \alpha V_{N,K}(x_1, \ldots, x_N; R_1, \ldots, R_k)\,. \tag{2.1}$$

The Pauli exclusion principle for fermions dictates that $H_{N,K}$ acts on functions in the anti-symmetric N-fold tensor product $\wedge^N L^2(\mathbb{R}^3; \mathbb{C}^q)$. We use units in which $\hbar = c = 1$, $\alpha > 0$ is the fine structure constant, and

$$\begin{aligned} V_{N,K}(x_1, \ldots, x_N; R_1, \ldots R_K) := & \sum_{1 \leq i < j \leq N} |x_i - x_j|^{-1} - \sum_{j=1}^{N} \sum_{k=1}^{K} Z_k |x_j - R_k|^{-1} \\ & + \sum_{1 \leq k < l \leq K} Z_k Z_l |R_k - R_l|^{-1}. \end{aligned} \tag{2.2}$$

is the Coulomb potential (electron-electron, electron-nuclei, nuclei-nuclei, respectively). In this model there is no interaction of the electron *spin* with the magnetic field. Note that we absorb the electron charge $\sqrt{\alpha}$ into the vector potential A, i.e., we write $A(x)$ instead of $\sqrt{\alpha} A(x)$ in (2.1). Since A is arbitrary and our bounds are independent of A, this does not affect our results.

Stability of matter means that $H_{N,K}$ is bounded from below by a constant times $(N+K)$, independently of the positions R_k of the nuclei and of A. For a thorough discussion see [L1, L2]. By scaling all spatial coordinates it is easy to see that either $\inf_{R_k,A}(\inf \operatorname{spec} H_{N,K}) = -mN$ or $= -\infty$.

We shall prove the following.

Theorem 2.1 (Stability of relativistic matter with magnetic fields). *For $q\alpha \leq 1/66.5$ and $\alpha Z_j \leq 2/\pi$ for all j,*

$$H_{N,K} \geq -mN$$

for all N, K, $R_1, \ldots, R_K$ and A.

For electrons $q = 2$ and hence our proof works up to

$$\alpha = \frac{1}{133} > \frac{1}{137}\,.$$

The rest of this paper contains the proof of Theorem 2.1, but let us first state an obvious fact.

Corollary 2.2. *As a multiplication operator on $\wedge^N L^2(\mathbb{R}^3;\mathbb{C}^q)$,*

$$V_{N,K}(x_1,\ldots,x_N;R_1,\ldots R_K) \geq -\max\{66.5\,q,\ \pi Z_j/2\}\sum_{j=1}^{N}|p_j + A(x_j)| \tag{2.3}$$

for all A.

This, of course, is just a rewording of Theorem 2.1, but the point is that it provides a lower bound for the Coulomb potential of interacting particles in terms of a one-body operator $|p + A(x)|$. This operator is dominated by the *nonrelativistic* operator $|p + A(x)|^2$ and, therefore, (2.3) is useful in certain nonrelativistic problems. For example, an inequality of this type was used in [LLoSo] to prove stability of matter with the Pauli operator $|p+A(x)|^2 + \sigma\cdot B(x)$ in place of $|p+A(x)|^2$. It was also used in [LSiSo] to control the no-pair Brown-Ravenhall relativistic model.

An examination of the proof of Theorem 2 in [LY] shows that there are two places that do not permit the inclusion of a magnetic vector potential A. These are Theorem 9 (Localization of kinetic energy – general form) and Theorem 11 (Lower bound to the short-range energy in a ball). Our Theorem 3.1 is precisely the extension of Theorem 9 to the magnetic case. It may be regarded as a diamagnetic inequality on the localization error. It implies that Theorem 10 in [LY] holds also in the magnetic case, without change except for replacing $|p|$ by $|p + A|$; see Theorem 3.2 below.

A substitute for Theorem 11 in [LY] will be given in Theorem 4.5 below. It is based on the observation that an estimate on eigenvalue sums of a non-magnetic operator with discrete spectrum implies a similar estimate (with a modified constant) for the corresponding magnetic operator. This is not completely obvious, since there is no diamagnetic inequality for *sums* of eigenvalues. (In fact, a conjectured diamagnetic inequality actually fails for fermions on a lattice and leads to the 'flux phase' [L3].) It

is for the different constants in Theorem 11 in [LY] and in our Theorem 4.5 that our bound on α_c become worse than the one in [LY].

As should be clear from the above discussion, our main tool will be a diamagnetic inequality for single functions. The one we use is the diamagnetic inequality for the heat kernel. In the relativistic case it states that for any $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$ and $u \in L^2(\mathbb{R}^3)$ one has

$$\big|\big(\exp(-t|p+A|)u\big)(x)\big| \leq \big(\exp(-t|p|)|u|\big)(x), \quad x \in \mathbb{R}^3. \tag{2.4}$$

This follows with the help of the subordination formula

$$e^{-|\xi|} = \int_0^\infty e^{-t-|\xi|^2/(4t)} \, \frac{dt}{\sqrt{\pi t}}$$

from the 'usual' (nonrelativistic) diamagnetic inequality for the semigroup $\exp(-t|p+A|^2)$; see, e.g., [S3]. The heat kernel is not prominent in [LY], and our reformulation of some of the key estimates in [LY] in terms of the heat kernel is the principal novel feature of this paper.

3. Localization of the kinetic energy with magnetic fields

3.1. **Relativistic IMS formula.** In this subsection we establish the analogue of Theorem 9 in [LY] in the general case $A \neq 0$. First, recall that the IMS formula in the nonrelativistic case says that for any u and A

$$\int_{\mathbb{R}^3} |(p+A)u|^2 \, dx = \sum_{j=0}^n \int_{\mathbb{R}^3} |(p+A)(\chi_j u)|^2 \, dx - \int_{\mathbb{R}^3} \sum_{j=0}^n |\nabla \chi_j|^2 |u|^2 \, dx$$

whenever χ_j are real functions with $\sum_{j=1}^n \chi_j^2 \equiv 1$. In this case the localization error $\sum_{j=0}^n |\nabla \chi_j|^2$ is local and independent of A. The analogue in the relativistic case is the following special case of [FLSe, Lemma B.1]. For the sake of completeness, we include its proof here.

Theorem 3.1 (Localization of kinetic energy – general form). *Let $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)$. If $\chi_0, \ldots, \chi_n$ are real Lipschitz continuous functions on $\mathbb{R}^3$ satisfying $\sum_{j=0}^n \chi_j^2 \equiv 1$, then one has*

$$(u, |p+A|u) = \sum_{j=0}^n (\chi_j u, |p+A|\chi_j u) - (u, L_A u)\,. \tag{3.1}$$

Here L_A is a bounded operator with integral kernel

$$L_A(x,y) := k_A(x,y) \sum_{j=0}^n (\chi_j(x) - \chi_j(y))^2\,,$$

where $k_A(x,y) := \lim_{t\uparrow 0} t^{-1} \exp(-t|p+A|)(x,y)$ for a.e. $x, y \in \mathbb{R}^3$ and

$$|k_A(x,y)| \leq \frac{1}{2\pi^2 |x-y|^4}\,. \tag{3.2}$$

Note that (3.2) says that

$$|L_A(x,y)| \leq L(x,y) := \frac{1}{2\pi^2|x-y|^4} \sum_{j=0}^{n} (\chi_j(x) - \chi_j(y))^2 \,. \tag{3.3}$$

Here, $L(x,y)$ is the same as in [LY, Eq. (3.7)]. Therefore, (3.2) is a diamagnetic inequality for the localization error.

Proof. We write $k_A(x,y,t) := \exp(-t|p+A|)(x,y)$ for the heat kernel and find

$$\sum_{j=0}^{n} (\chi_j u, (1-\exp(-t|p+A|))\chi_j u) = (u, (1-\exp(-t|p+A|))u)$$
$$+ \frac{1}{2} \sum_{j=0}^{n} \iint k_A(x,y,t)(\chi_j(x) - \chi_j(y))^2 \overline{u(x)} u(y) \, dx \, dy \,.$$

(This is proved simply by writing out both sides in terms of $k_A(x,y,t)$ and using $\sum \chi_j^2 \equiv 1$.) Now we divide by t and let $t \to 0$. The left side converges to $\sum_{j=0}^{n} (\chi_j u, |p+A|\chi_j u)$. Similarly, the first term on the right side divided by t converges to $(u, |p+A|u)$. Hence the last term divided by t converges to some limit $(u, L_A u)$. The diamagnetic inequality (2.4) says that

$$|k_A(x,y,t)| \leq \exp(-t|p|)(x,y) = \frac{t}{\pi^2 \left(|x-y|^2 + t^2\right)^2}$$

(see [LLo, Eq. 7.11(9)]). This implies, in particular, that L_A is a bounded operator. Now it is easy to check that L_A is an integral operator and that the absolute value of its kernel is bounded pointwise by the one of L in (3.3). ■

3.2. **Localization of the kinetic energy.** In this subsection we will bound the localization error L_A by a potential energy correction and an additive constant. This is the extension of Theorem 10 in [LY] to the case $A \neq 0$. It is important that both error terms in our bound can be chosen independently of A.

First we need to introduce some notation. We write

$$\mathcal{B}_R := \{x \,:\, |x| < R\}$$

for the ball of radius R and $\chi_{\mathcal{B}_R}$ for its characteristic function. If $R = 1$, we omit the index in the notation. We fix a constant $0 < \sigma < 1$ and Lipschitz continuous functions χ_0, χ_1 with $\chi_0^2 + \chi_1^2 \equiv 1$ such that $\operatorname{supp} \chi_1 \subset \overline{\mathcal{B}_{1-\sigma}}$. With these we define L as in (3.3) with $n = 1$. We decompose L in a short-range part L^0 and a long-range part L^1 given by the kernels

$$L^1(x,y) := L(x,y)\chi_{\mathcal{B}}(x)\chi_{\mathcal{B}}(y)\chi_{\mathcal{B}_\sigma}(x-y), \qquad L^0(x,y) := L(x,y) - L^1(x,y). \tag{3.4}$$

Define

$$\Omega := \frac{1}{2} \operatorname{Tr} \left(L^0\right)^2 \tag{3.5}$$

and, for an arbitrary positive function h on $\mathcal{B}$,

$$\theta(x) := h^{-1}(x) \int_{\mathcal{B}} L^1(x,y) h(y)\, dy = h^{-1}(x) \chi_{\mathcal{B}}(x) \int_{|y|<1,\, |x-y|<\sigma} L(x,y) h(y)\, dy\,.$$

Finally, for $\varepsilon > 0$ we define the function

$$U^*_\varepsilon := \varepsilon \chi_{\mathcal{B}_{1-\sigma}} + \theta \tag{3.6}$$

and note that U^*_ε is supported in $\overline{\mathcal{B}}$.

Theorem 3.2 (Localization of kinetic energy – explicit bound in the one-center case). *For any $\varepsilon > 0$ and any non-negative trace-class operator γ one has*

$$\operatorname{Tr} \gamma |p+A| \geq \sum_{j=0}^{1} \operatorname{Tr} \chi_j \gamma \chi_j (|p+A| - U^*_\varepsilon) - \varepsilon^{-1} \Omega \|\gamma\|. \tag{3.7}$$

For $A = 0$ this is exactly Theorem 10 in [LY]. As explained there, U^*_ε is a potential energy correction with only slightly larger support than χ_1. The last term in (3.7) is due to the long range nature of $|p+A|$. It depends on γ through its norm $\|\gamma\|$ but not through its trace. We emphasize again that both error terms in the inequality (3.7) are independent of A.

Proof. The localization formula (3.1) yields

$$\operatorname{Tr} \gamma |p+A| = \sum_{j=0}^{1} \operatorname{Tr} \chi_j \gamma \chi_j |p+A| - \operatorname{Tr} \gamma L_A,$$

so we only have to find an upper bound for $\operatorname{Tr} \gamma L_A$. We decompose $L_A = L_A^0 + L_A^1$ in the manner of (3.4) and, following the proof of Theorem 10 in [LY] word by word, we obtain

$$\operatorname{Tr} \gamma L_A^0 \leq \varepsilon \operatorname{Tr} \gamma \chi_{\mathcal{B}_{1-\sigma}} + (2\varepsilon)^{-1} \|\gamma\| \operatorname{Tr} \left(L_A^0\right)^2, \qquad \operatorname{Tr} \gamma L_A^1 \leq \operatorname{Tr} \gamma \theta_A.$$

Here $\theta_A(x) := 0$ if $x \notin \mathcal{B}$ and, if $x \in \mathcal{B}$,

$$\theta_A(x) := h^{-1}(x) \int_{\mathcal{B}} |L_A^1(x,y)| h(y)\, dy.$$

The estimate $|L_A(x,y)| \leq L(x,y)$ from Theorem 3.1 implies that $\operatorname{Tr} \left(L_A^0\right)^2 \leq 2\Omega$ and that $\theta_A \leq \theta$. This leads to the stated lower bound. ∎

4. Bounds on eigenvalues in balls

So far we have considered $|p+A|$ and its heat kernel. Now we address $|p+A| - 2/(\pi|x|)$ and its heat kernel. First of all, let us recall Kato's inequality [Ka, Eq. (V.5.33)]

$$(u, |p| u) \geq (2/\pi)(u, |x|^{-1} u)\,. \tag{4.1}$$

(See also [H, W, KPS].)

Now let $\Gamma \subset \mathbb{R}^3$ be an open set (we shall be interested in the case where Γ is a ball) and consider the quadratic form given by $Q_\Gamma(u) = (u, (|p| - 2/\pi|x|)u)$, restricted to those functions $u \in L^2(\mathbb{R}^3)$ that satisfy $u = 0$ on Γ^c, the complement of Γ. Of course, we also require u to be in the quadratic form domain of $|p| - 2/\pi|x|$. The quadratic form Q_Γ is non-negative by (4.1) and it is closed (because the form $|p| - 2/\pi|x|$ is closed on $L^2(\mathbb{R}^3)$ and limits of functions that are zero on Γ^c are zero on Γ^c). From this it follows that there is a self-adjoint operator H_Γ on some domain in $L^2(\Gamma)$ such that $Q_\Gamma(u) = (u, H_\Gamma u)$. With this operator, we can define the 'heat kernel' $\exp(-tH_\Gamma)$ on $L^2(\Gamma)$ and its trace. (The fact that the trace is finite when the volume of Γ is finite follows from subsequent considerations.)

Similarly, for a magnetic vector potential $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$, we define the operator H_Γ^A in $L_2(\Gamma)$ using the quadratic form $(u, (|p+A| - 2/\pi|x|)u)$. Note that (2.4) implies that

$$(u, |p+A|u) \geq (|u|, |p||u|)\,. \tag{4.2}$$

This, together with (4.1), shows that $(u, (|p+A| - 2/\pi|x|)u)$ is non-negative.

Lemma 4.1 (**Heat kernel diamagnetic inequality).** *Let* $\Gamma \subset \mathbb{R}^3$ *and let* $A \in L^2_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{R}^3)$. *Then, for any* $t > 0$,

$$\mathrm{Tr}_{L^2(\Gamma)} \exp\left(-tH_\Gamma^A\right) \leq \mathrm{Tr}_{L^2(\Gamma)} \exp\left(-tH_\Gamma\right)\,. \tag{4.3}$$

Proof. For $n = 0, 1, 2, \ldots$ let $h_n := |p| - 2/(\pi|x|) + n\chi_{\Gamma^c}$ in $L^2(\mathbb{R}^3)$, where χ_{Γ^c} denotes the characteristic function of the complement of Γ. Similarly, let $h_n^A := |p+A| - 2/(\pi|x|) + n\chi_{\Gamma^c}$. The diamagnetic inequality (2.4) and standard approximation arguments using Trotter's product formula imply that, for any $u \in L^2(\mathbb{R}^3)$,

$$\left|\left(\exp(-th_n^A)u\right)(x)\right| \leq \left(\exp(-th_n)|u|\right)(x)\,.$$

(See [FLSe, Section 6.2] for details of the argument.)

By the monotone convergence theorem [S1, Thm. 4.1], $\exp(-th_n)$ converges strongly to $\exp(-tH_\Gamma)$ on the subspace $L^2(\Gamma)$, and similarly for h_n^A. It follows that, for any $u \in L^2(\Gamma)$,

$$\left|\left(\exp(-tH_\Gamma^A)u\right)(x)\right| \leq \left(\exp(-tH_\Gamma)|u|\right)(x)\,.$$

Theorem 2.13 in [S3] yields the inequality $\|\exp(-tH_\Gamma^A)\|_2 \leq \|\exp(-tH_\Gamma)\|_2$ for the Hilbert-Schmidt norm, and hence $\|\exp(-2tH_\Gamma^A)\|_1 \leq \|\exp(-2tH_\Gamma)\|_1$ for the trace norm by the semigroup property. This holds for all $t > 0$, and hence proves (4.3). ∎

We use the notation $(x)_- = \max\{0, -x\}$ for the negative part of $x \in \mathbb{R}$ in the following.

Lemma 4.2. *Assume that there is constant* $M > 0$ *such that*

$$\mathrm{Tr}_{L^2(\Gamma)} \left(H_\Gamma - \Lambda\right)_- \leq M\Lambda^4 \tag{4.4}$$

for all $\Lambda \geq 0$. *Then*

$$\mathrm{Tr}_{L^2(\Gamma)} \left(H_\Gamma^A - \Lambda\right)_- \leq \frac{6e^3}{4^3} M\Lambda^4 \tag{4.5}$$

for all $\Lambda \geq 0$.

We note the the numerical factor in (4.5) equals $6(e/4)^3 \approx 1.883$. This factor is the price we have to pay, using our methods, to include an arbitrary magnetic field. It is the reason of the decrease of α_c from 1/94 to 1/133.

Proof. Since $(x)_- \leq e^{-x-1}$, we have

$$\mathrm{Tr}_{L^2(\Gamma)} \left(H_\Gamma^A - \Lambda\right)_- \leq \frac{e^{t\Lambda}}{te} \mathrm{Tr}_{L^2(\Gamma)} \exp\left(-tH_\Gamma^A\right)$$

for any $t > 0$. Using the diamagnetic inequality (4.3),

$$\mathrm{Tr}_{L^2(\Gamma)} \exp\left(-tH_\Gamma^A\right) \leq \mathrm{Tr}_{L^2(\Gamma)} \exp\left(-tH_\Gamma\right) .$$

Moreover, integrating by parts twice, $e^{-tx} = t^2 \int_0^\infty e^{-t\lambda}(x-\lambda)_- \, d\lambda$, and hence

$$\mathrm{Tr}_{L^2(\Gamma)} \exp\left(-tH_\Gamma\right) = t^2 \int_0^\infty e^{-t\lambda}\, \mathrm{Tr}_{L^2(\Gamma)} \left(H_\Gamma - \lambda\right)_- \, d\lambda \,.$$

Using the assumption (4.4), we thus obtain

$$\mathrm{Tr}_{L^2(\Gamma)} \left(H_\Gamma^A - \Lambda\right)_- \leq \frac{te^{t\Lambda}}{e} M \int_0^\infty e^{-t\lambda} \lambda^4 d\lambda = 24 \frac{e^{t\Lambda}}{t^4 e} M \,.$$

To minimize the right side, the optimal choice of t is $t = 4/\Lambda$. This yields (4.5). ∎

In [LY, Thm. 11] it is shown that (4.4) holds for $\Gamma = \mathcal{B}_R$ a ball of radius R centered at the origin. More precisely, the following proposition holds.

Proposition 4.3. *For any* $R > 0$ *and* $\Lambda \geq 0$,

$$\mathrm{Tr}_{L^2(\mathcal{B}_R)} \left(H_{\mathcal{B}_R} - \Lambda\right)_- \leq 4.4827\, R^3 \Lambda^4 \,.$$

Proposition 4.3 follows from Theorem 11 in [LY] by choosing χ to be the characteristic function of the ball $\mathcal{B}_R$, $q = 1$ and γ to be the projection onto the negative spectral subspace of $H_{\mathcal{B}_R} - \Lambda$.

Remark 4.4. It is illustrative to compare Proposition 4.3 with the Berezin-Li-Yau type bound

$$\mathrm{Tr}_{L^2(\Gamma)} \left(|p|_\Gamma - \Lambda\right)_- \leq \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_\Gamma \left(|\xi| - \Lambda\right)_- \, dx\, d\xi = \frac{1}{24\pi^2} \Lambda^4 |\Gamma| \,. \tag{4.6}$$

(This can be proved in the same way as [LLo, Thm. 12.3].) The right side of (4.6) is the semi-classical phase-space integral. The operator $|p|_\Gamma$ is defined as H_Γ above, but without the Hardy-term $2/(\pi|x|)$. If the Hardy term were added, the phase-space integral would diverge (provided Γ contains the origin), but Proposition 4.3 says that a bound of the form (4.6) still holds. (An examination of the proof in [LY] shows that Proposition 4.3 actually holds for any open set Γ of finite measure.)

Combining Lemma 4.2 and Proposition 4.3 we obtain the following theorem, which replaces [LY, Thm. 11] in the magnetic case.

Theorem 4.5 (Lower bound on the short-range energy in a ball). *Let $C > 0$ and $R > 0$ and let*

$$H^A_{CR} := |p + A| - \frac{2}{\pi|x|} - \frac{C}{R}$$

be defined on $L^2(\mathbb{R}^3)$ as a quadratic form. Let $0 \le \gamma \le q$ be a density matrix (i.e., a positive trace-class operator) and let χ by any bounded function with support in $\mathcal{B}_R$. Then

$$\operatorname{Tr} \overline{\chi}\gamma\chi H^A_{CR} \ge -8.4411 \frac{qC^4}{R} \|\chi\|^2_\infty . \tag{4.7}$$

As compared with [LY, Thm. 11], the constant has been multiplied by $6(e/4)^3$, and $\|\chi\|^2_\infty$ appears instead of $|\mathcal{B}_R|^{-1}\|\chi\|^2_2$.

Proof. Note that

$$\operatorname{Tr} \overline{\chi}\gamma\chi H^A_{CR} = \operatorname{Tr} \overline{\chi}\gamma\chi \left(H^A_{\mathcal{B}_R} - C/R\right) \ge -\|\overline{\chi}\gamma\chi\|_\infty \operatorname{Tr}_{L^2(\mathcal{B}_R)}(H^A_{\mathcal{B}_R} - C/R)_- .$$

The assertion follows from Lemma 4.2 and Proposition 4.3, observing that $\|\overline{\chi}\gamma\chi\|_\infty \le q\|\chi\|^2_\infty$. ∎

5. Proof of Theorem 2.1

We assume that the reader is familiar with the proof of Theorem 2 in [LY]. We shall only emphasize changes in their argument. The main idea is to replace Theorems 10 and 11 in [LY] by our Theorems 3.2 and 4.5, respectively.

There are some immediate simplifications. First, in view of the simple inequality $\sqrt{|p|^2 + m^2} \ge |p|$ it is enough to prove the Theorem 2.1 for $m = 0$. Moreover, by the convexity argument of [DL] it suffices to treat the case $Z_1 = \ldots = Z_K =: z$ and $\alpha z = 2/\pi$. So henceforth we assume $m = 0$, $Z_1 = \ldots = Z_K = z$ and $\alpha z = 2/\pi$.

Let $D_k := \min\{|R_k - R_l| : l \ne k\}$ and define the Voronoi cell

$$\Gamma_k := \{x \in \mathbb{R}^3 : |x - R_k| < |x - R_l| \text{ for all } l \ne k\}.$$

Fix $0 < \lambda < 1$ and define a function $W := G + F$ in each Voronoi cell by

$$G(x) := z|x - R_k|^{-1}, \qquad F(x) := D_k^{-1}\tilde{F}(|x - R_k|/D_k), \qquad x \in \Gamma_k,$$

where

$$\tilde{F}(t) := \begin{cases} 2^{-1}(1 - t^2)^{-1} & \text{if } t \le \lambda, \\ (\sqrt{2z} + \frac{1}{2})t^{-1} & \text{if } t > \lambda. \end{cases}$$

By the electrostatic inequality in [LY, Sect. III, Step A] our Theorem 2.1 will follow if we can prove that

$$\operatorname{Tr} \gamma(|p + A| - \alpha W) \ge -\frac{z^2\alpha}{8} \sum_{k=1}^{K} D_k^{-1} \tag{5.1}$$

for some $0 < \lambda < 1$ and all density matrices γ with $0 \le \gamma \le q$). Note that (5.1) is an inequality for a *one-particle* operator.

For fixed $0 < \sigma < 1/3$ we choose χ, h as in (3.22), (3.24) in [LY]. Note that $\operatorname{supp}\chi \subset \overline{\mathcal{B}_{1-\sigma}}$. Let

$$\chi_k(x) := \chi(|x-R_k|/D_k), \qquad h_k(x) := h(|x-R_k|/D_k).$$

After scaling and translation, Proposition 3.2 yields that for any $0 \le \gamma \le q$

$$\begin{aligned} \operatorname{Tr}\gamma(|p+A|-\alpha W) \ge & \operatorname{Tr}\chi_1\gamma\chi_1(|p+A|-U^*_{1,\varepsilon}-\alpha W) \\ & + \operatorname{Tr}(1-\chi_1^2)^{1/2}\gamma(1-\chi_1^2)^{1/2}(|p+A|-U^*_{1,\varepsilon}-\alpha W) \\ & - \varepsilon^{-1}q\Omega/D_1 \,. \end{aligned} \tag{5.2}$$

Here, $U^*_{1,\varepsilon}(x) := D_1^{-1}U^*_\varepsilon((x-R_1)/D_1)$ and Ω, U^*_ε were defined in (3.5), (3.6). (Note that our Ω is denoted by Ω_1 in [LY]). Recall that U^*_ε and Ω are independent of A.

We turn to the first term on the right side of (5.2). Let C be a constant such that

$$C \ge (1-\sigma)\left(\alpha\tilde F(|x|) + U^*_\varepsilon(x)\right) \qquad \text{for } |x| \le 1-\sigma \,. \tag{5.3}$$

Note that χ_1 is supported on a ball of radius $(1-\sigma)D_1$ centered at R_1. Hence $\alpha W(x) = (2/\pi)|x-R_1|^{-1} + D_1^{-1}\tilde F(|x-R_1|/D_1)$ on the support of χ_1 and we can apply Theorem 4.5 to obtain the lower bound

$$\begin{aligned} \operatorname{Tr}\chi_1\gamma\chi_1(|D-A|-U^*_{1,\varepsilon}-\alpha W) \ge & \operatorname{Tr}\chi_1\gamma\chi_1\left(|p+A| - \frac{2}{\pi|x-R_1|} - \frac{C}{(1-\sigma)D_1}\right) \\ \ge & -8.4411\frac{qC^4}{(1-\sigma)D_1} \,. \end{aligned} \tag{5.4}$$

We used also that $|\chi_1| \le 1$. Inserting (5.4) into (5.2) we find

$$\begin{aligned} \operatorname{Tr}\gamma(|p+A|-\alpha W) \ge & -qD_1^{-1}\tilde A \\ & + \operatorname{Tr}(1-\chi_1^2)^{1/2}\gamma(1-\chi_1^2)^{1/2}(|p+A|-U^*_{1,\varepsilon}-\alpha W) \end{aligned}$$

with

$$\tilde A := \frac{\Omega}{\varepsilon} + 8.4411\frac{C^4}{(1-\sigma)} \,.$$

This estimate is exactly of the form (3.26) in [LY], except for the value of the constant in $\tilde A$ (which is called A in [LY]). Starting from there one can continue along the lines of their proof. We need only note that in order to bound the last term in (3.29) in [LY] one uses the Daubechies inequality [D], which holds with the same constant in the presence of a magnetic field. (This is explained, for instance, in [LLoSi, Sect. 5].) We conclude that stability holds as long as

$$\alpha q(\tilde A + J) \le \frac{1}{2\pi^2} \,, \tag{5.5}$$

where, as in [LY, Eq. (3.31)],

$$J := 0.0258 \int_{|x|\ge 1-3\sigma} \left[\frac{2}{\pi|x|} + \alpha\tilde F(|x|) + U^*_\varepsilon(x)\right]^4 dx \,.$$

This completes our proof of Theorem 2.1, except for our bound on the critical α, which we justify now.

As in [LY], we choose $\sigma = 0.3$, $\varepsilon = 0.2077$ and $\lambda = 0.97$. Our goal is to prove stability when $q\alpha \le 1/66.5$. We may assume $\alpha < 1/47$, which is the assumption used in [LY]. Hence we can use the estimate $J \le 1.64$ from [LY, Eq. (3.40)].

To bound $\tilde{A}$, note that $\varepsilon^{-1}\Omega = 0.5571$ as in [LY, Eq. (3.30)]. It remains to choose an appropriate C satisfying (5.3). For $|x| \le 0.7$ we have $|\tilde{F}(|x|)| \le 1/1.02$. Moreover, for U_ε^* we use the same estimate as in [LY], namely $U_\varepsilon^*(x) \le 0.2077 + 0.5751 = 0.7828$. Using $\alpha \le 1/(66.5\,q) \le 1/66.5$, (5.3) therefore holds with

$$0.7\,(1/(66.5 \cdot 1.02) + 0.7828) < 0.5583 =: C\,.$$

This leads to a value of $\tilde{A} = 1.7287$. Hence (5.5) holds for $q\alpha \le 1/66.5$.

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